Properties of Logarithms

Memorize the following properties:

$y = \log_a x$ if and only if $a^y = x$

$\log_a 1 = 0$ $\quad\quad$ $\log_a a = 1$ $\quad\quad$ $a^{\log_a M} = M$ $\quad\quad$ $\log_a a^r = r$

The Log of a Product Equals the Sum of the Logs: $\log_a (MN) = \log_a M + \log_a N$

The Log of a Quotient Equals the Difference of the Logs: $\log_a \left(\frac{M}{N}\right) = \log_a M - \log_a N$

The Log of a Power Equals the Product of the Power and the Log: $\log_a M^r = r \log_a M$

If $M = N$, then $\log_a M = \log_a N$.

If $\log_a M = \log_a N$, then $M = N$.

Change of Base Formula: $\log_a M = \dfrac{\log_b M}{\log_b a}$ $\quad\quad$ $\log_a M = \dfrac{\log M}{\log a}$ $\quad\quad$ $\log_a M = \dfrac{\ln M}{\ln a}$

Examples: Find the exact value of each expression. (Do not use a calculator).

a) $\log_{0.6} 0.6^{-3.2} = \boxed{-3.2}$

"The exponent to which you must raise 0.6 to get $0.6^{-3.2}$"

b) $5^{\log_5 3} = \boxed{3}$

$\log_5 3$ is the exponent to which you must raise 5 to get 3. When you raise 5 to that exponent, you get 3.

c) $\log_7 7^{-1} = \boxed{-1}$

d) $e^{\ln 12} = \boxed{12}$

Examples: Use properties of logarithms to find the exact value of each expression. (Do not use a calculator).

a) $\log_6 18 - \log_6 3$

$= \log_6 \left(\frac{18}{3}\right)$

$= \log_6 6 = \boxed{1}$

b) $5^{\log_5 6 + \log_5 7}$

$= 5^{\log_5 (6\cdot 7)}$

$= 5^{\log_5 42}$

$= \boxed{42}$

c) $e^{\log_{e^2} 9} = e^a$

let $a = \log_{e^2} 9$.

$(e^2)^a = 9$

$e^{2a} = 9$

$\ln 9 = 2a$

$a = \frac{\ln 9}{2}$

$a = \ln 9^{1/2}$

$a = \ln 3$

$e^a = e^{\ln 3}$

$= \boxed{3}$

d) $\log_3 8 \cdot \log_8 9 =$

$\log_8 9^{\log_3 8}$

$=$

$\log_8 (3^2)^{\log_3 8}$

$=$

$\log_8 3^{2\log_3 8} = \log_8 3^{\log_3 8^2}$

$= \log_8 8^2$

$= \boxed{2}$

Examples: Write each expression as a sum/difference of logarithms. Express powers as factors.

a) $\log_7 (x^5)$

$= \boxed{5 \log_7 x}$

b) $\ln (xe^x)$

$= \ln x + \ln e^x$

$= \boxed{\ln x + x}$

c) $\log_2\left(\frac{a}{b^2}\right),\ a>0,\ b>0$

$= \log_2 a - \log_2 b^2$

$= \boxed{\log_2 a - 2\log_2 b}$

d) $\ln\left[\frac{(x-4)^2}{x^2-1}\right]^{2/3}$; $x>4$ $= \frac{2}{3}\ln\left[\frac{(x-4)^2}{(x+1)(x-1)}\right]$

$= \frac{2}{3}\left[\ln(x-4)^2 - \ln[(x+1)(x-1)]\right]$

$= \frac{2}{3}\left[2\ln(x-4) - [\ln(x+1)+\ln(x-1)]\right]$

$= \frac{2}{3}\left[2\ln(x-4) - \ln(x+1) - \ln(x-1)\right]$

$= \boxed{\frac{4}{3}\ln(x-4) - \frac{2}{3}\ln(x+1) - \frac{2}{3}\ln(x-1)}$

Examples: Write each expression as a single logarithm.

a) $3\log_5 u + 4\log_5 v =$

$\log_5 u^3 + \log_5 v^4 =$

$\boxed{\log_5(u^3v^4)}$

b) $\log_4(x^2-1) - 5\log_4(x+1) =$

$\log_4(x^2-1) - \log_4(x+1)^5 =$

$\log_4\left(\frac{x^2-1}{(x+1)^5}\right) = \log_4\left(\frac{(x-1)(x+1)}{(x+1)^5}\right)$

$= \boxed{\log_4\left(\frac{x-1}{(x+1)^4}\right)}$

c) $\log\left(\frac{x^2-2x-3}{x^2-4}\right) - \log\left(\frac{x^2+7x+6}{x+2}\right) =$

$\log\left(\frac{\frac{x^2-2x-3}{x^2-4}}{\frac{x^2+7x+6}{x+2}}\right) = \log\left(\frac{\frac{(x-3)(x+1)}{(x+2)(x-2)}}{\frac{(x+6)(x+1)}{x+2}}\right) =$

$\log\left(\frac{(x-3)\cancel{(x+1)}}{\cancel{(x+2)}(x-2)} \cdot \frac{\cancel{x+2}}{(x+6)\cancel{(x+1)}}\right) =$

$\log\left(\frac{x-3}{(x-2)(x+6)}\right) = \boxed{\log\left(\frac{x-3}{x^2+4x-12}\right)}$

d) $21\log_3\sqrt[3]{x} + \log_3(9x^2) - \log_3 9 =$

$\log_3(\sqrt[3]{x})^{21} + \log_3(9x^2) - \log_3 9 =$

$\log_3 x^7 + \log_3(9x^2) - \log_3 9 =$

$\log_3\left(\frac{x^7 \cdot 9x^2}{9}\right) = \log_3\left(\frac{9x^9}{9}\right)$

$= \boxed{\log_3 x^9}$

e) $\frac{1}{3}\log(x^3+1) + \frac{1}{2}\log(x^2+1) =$

$\log(x^3+1)^{1/3} + \log(x^2+1)^{1/2} =$

$\boxed{\log\left[(x^3+1)^{1/3}(x^2+1)^{1/2}\right]}$

Examples: Use the change of base formula to evaluate each logarithm.

a) $\log_6 9 =$

$\frac{\log 9}{\log 6}$ or $\frac{\ln 9}{\ln 6}$

$= \boxed{1.226}$

b) $\log_{\sqrt{2}} 7 =$

$\frac{\log 7}{\log\sqrt{2}}$ or $\frac{\ln 7}{\ln\sqrt{2}}$

$= \boxed{5.615}$

c) $\log_\pi \sqrt{3} =$

$\frac{\log\sqrt{3}}{\log\pi}$ or $\frac{\ln\sqrt{3}}{\ln\pi}$

$= \boxed{0.480}$

d) $\log_3 5 =$

$\frac{\log 5}{\log 3}$ or $\frac{\ln 5}{\ln 3}$

$= \boxed{1.465}$

Math 1050 – 6.6 Notes

Logarithmic and Exponential Equations

Solving Logarithmic Equations

- To avoid extraneous solutions, determine the domain of the variable first.
- Use the properties of logarithms to manipulate the equations.
- Try rewriting as an exponential function: $y = \log_a x \Leftrightarrow x = a^y$
- Remember the property: $\log_a M = \log_a N \Leftrightarrow M = N$

Examples:

a) $\log_3(3x-1) = 2$

$3x-1>0$
$3x>1$
$x>1/3$

$3^2 = 3x-1$
$9 = 3x-1$
$10 = 3x$
$\boxed{x = 10/3}$

b) $-2\log_4 x = \log_4 9$ $\quad x>0$

$\log_4 x^{-2} = \log_4 9$
$x^{-2} = 9$
$\frac{1}{x^2} = 9$
$9x^2 = 1$
$x^2 = 1/9$
$x = \pm\sqrt{1/9}$
$x = \pm 1/3$

Since $x>0$, we can throw out $-1/3$.

$\boxed{x = 1/3}$

c) $3\log_2(x-1) + \log_2 4 = 5$

$x-1>0$
$x>1$

$\log_2 (x-1)^3 + \log_2 4 = 5$
$\log_2[4(x-1)^3] = 5$
$2^5 = 4(x-1)^3$
$32 = 4(x-1)^3$
$(x-1)^3 = 8$
$\sqrt[3]{(x-1)^3} = \sqrt[3]{8}$
$x-1 = 2$
$\boxed{x = 3}$

d) $\ln(x+1) - \ln(x) = 2$

$x+1>0$ $\quad x>0$
$x>-1$

$\ln\left(\frac{x+1}{x}\right) = 2$
$e^2 = \frac{x+1}{x}$
$e^2x = x+1$
$e^2x - x = 1$
$x(e^2-1) = 1$
$\boxed{x = \frac{1}{e^2-1}}$

e) $\log_6(x+4) + \log_6(x+3) = 1$

$x+4>0$ $\quad x+3>0$
$x>-4$ $\quad x>-3$

$\log_6[(x+4)(x+3)] = 1$
$6^1 = (x+4)(x+3)$
$6 = x^2+7x+12$
$x^2+7x+6 = 0$
$(x+6)(x+1) = 0$
$x = \cancel{-6}, -1$
$\boxed{x = -1}$

f) $\log_a x + \log_a(x-2) = \log_a(x+4)$

$x>0$ $\quad x-2>0$ $\quad x+4>0$
$x>2$ $\quad x>-4$

$\log_a[x(x-2)] = \log_a(x+4)$
$x(x-2) = x+4$
$x^2-2x = x+4$
$x^2-3x-4 = 0$
$(x-4)(x+1) = 0$
$x = 4, \cancel{-1}$
$\boxed{x = 4}$

Solving Exponential Equations

- If possible, make the bases the same, set exponents equal, and solve: $a^u = a^v \Leftrightarrow u = v$.
- If bases cannot be made the same, use algebraic techniques or rewrite as a log and use log properties.
- Remember the property: $M = N \Leftrightarrow \log_a M = \log_a N$
- If an exact solution cannot be found, use a graphing utility to obtain an approximate solution.

Examples:

a) $2^{-x} = 1.5$

$\log_2 1.5 = -x$

$\boxed{x = -\log_2 1.5}$

$x = -\dfrac{\log 1.5}{\log 2}$

$\boxed{x \approx -0.585}$

b) $0.3\left(4^{0.2x}\right) = 0.2$

$4^{0.2x} = \dfrac{0.2}{0.3}$

$4^{0.2x} = \dfrac{2}{3}$

$\log_4\left(\frac{2}{3}\right) = 0.2x$

$\boxed{x = \dfrac{\log_4(2/3)}{0.2}}$

$x = \left(\dfrac{\log(2/3)}{\log 4}\right) \div 0.2$

$\boxed{x \approx -1.462}$

c) $e^{x+3} = \pi^x$

$\log_e \pi^x = x+3$

$\ln \pi^x = x+3$

$x \ln \pi = x + 3$

$x \ln \pi - x = 3$

$x(\ln \pi - 1) = 3$

$\boxed{x = \dfrac{3}{\ln \pi - 1}}$ $\boxed{x \approx 20.728}$

d) $36^x - 6(6)^x = -9$

$36^x - 6(6)^x + 9 = 0$

$(6^2)^x - 6(6)^x + 9 = 0$

$6^{2x} - 6(6)^x + 9 = 0$

$u^2 - 6u + 9 = 0$

$(u-3)^2 = 0$

$u = 3$

$u = 6^x$

$3 = 6^x$

$\boxed{x = \log_6 3}$

$\boxed{x \approx 0.613}$

e) $2(49)^x + 11(7)^x + 5 = 0$

$2(7^2)^x + 11(7)^x + 5 = 0$

$2(7)^{2x} + 11(7)^x + 5 = 0$

$2u^2 + 11u + 5 = 0$

$(2u^2 + 10u) + (u+5) = 0$

$2u(u+5) + 1(u+5) = 0$

$(u+5)(2u+1) = 0$

$u = 7^x$

$u = -5$ or $u = -\frac{1}{2}$

$7^x = -5$ or $7^x = -\frac{1}{2}$

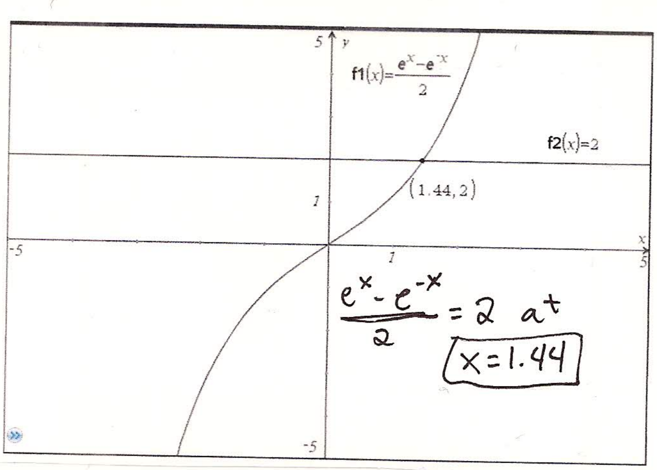
Raising 7 to a power will never result in a negative number.

$\boxed{\text{No solution}}$

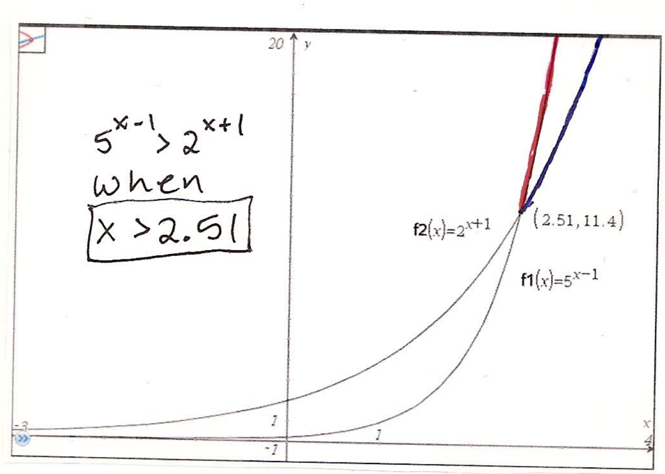
Solving by Graphing

1. Let each side of the equal sign be a separate function. Set one as Y_1 and the other as Y_2.
2. Graph Y_1 and Y_2 on your graphing calculator.
3. Find the intersection of the two graphs.
4. To solve an inequality, determine the interval of x-values that make the statement true.

a) $\dfrac{e^x - e^{-x}}{2} = 2$



b) $5^{x-1} > 2^{x+1}$



Compound Interest

Simple Interest: If a principal of P dollars is borrowed for a period of t years at a per annum interest rate r, expressed as a decimal, the interest I charged is $I = Prt$.

When the interest due at the end of a payment period is added to the principal so that the interest computed at the end of the next payment period is based on the new principal amount (old principal plus interest), the interest is said to have been **compounded. Compound interest** is interest paid on principal and previously earned interest.

Example: A credit union pays 7% per annum compounded quarterly on a certain savings plan. If \$900 is deposited in such a plan and the interest is left to accumulate, how much is in the account after 1 year? Hint: Find the simple interest each time the balance is compounded, adding the new interest each time.

1st Quarter:
$P = \$900$ $r = .07$ $t = .25$
$I = (900)(.07)(.25)$
$= \$15.75$
Amount $= \$915.75$

2nd Quarter:
$P = \$915.75$ $r = .07$ $t = .25$
$I = (915.75)(.07)(.25)$
$= \$16.03$
Amount $= \$915.75 + \$16.03$
$= \$931.78$

3rd Quarter:
$P = \$931.78$
$r = .07$ $t = .25$
$I = (931.78)(.07)(.25)$
$= \$16.31$
Amount $= \$931.78 + \$16.31$
$= \$948.09$

4th Quarter:
$P = \$948.09$
$r = .07$ $t = .25$
$I = (948.09)(.07)(.25)$
$= \$16.59$
Amount $=$
$\$948.09 + \$16.59$
$= \boxed{\$964.68}$

Compound Interest Formula
The amount A after t years due to a principal P invested at an annual interest rate t compounded n times per year is $A = P \cdot \left(1 + \frac{r}{n}\right)^{nt}$.

Example: Investing \$1000 at an annual rate of 9% compounded annually, semiannually, quarterly, monthly, and daily will yield the following amounts after 1 year:

Annual Compounding ($n = 1$): $A = 1000\left(1 + \frac{.09}{1}\right)^{(1 \cdot 1)} = \boxed{\$1090}$

Semiannual Compounding ($n = 2$): $A = 1000\left(1 + \frac{.09}{2}\right)^{(2 \cdot 1)} = \boxed{\$1092.03}$

Quarterly Compounding ($n = 4$): $A = 1000\left(1 + \frac{.09}{4}\right)^{(4 \cdot 1)} = \boxed{\$1093.08}$

Monthly Compounding ($n = 12$): $A = 1000\left(1 + \frac{.09}{12}\right)^{(12 \cdot 1)} = \boxed{\$1093.81}$

Daily Compounding ($n = 365$): $A = 1000\left(1 + \frac{.09}{365}\right)^{(365 \cdot 1)} = \boxed{\$1094.16}$

Continuous Compounding
The amount A after t years due to a principal P invested at an annual interest rate r compounded continuously is $A = Pe^{rt}$.

Example: Find the amount A that results from investing a principle P of \$1000 at an annual rate r of 9% compounded continuously for a time t of 1 year.

$A = (1000)e^{(.09 \cdot 1)} = \boxed{\$1094.17}$

Effective Rate of Interest: the equivalent annual simple rate of interest that would yield the same amount as compounding after 1 year.

Example: Compute the effective rate of interest for \$3000 with an interest of 6% per annum compounded continuously. $P = \$3000$ $r = .06$ $t = 1$

1. Find the amount resulting by compounding continuously: $A = Pe^{rt}$.
2. Find the Interest paid: $I = A - P$.
3. Find the interest rate of the interest paid using the simple interest formula: $I = Prt$.

1. $A = (3000)e^{(.06 \cdot 1)} = \$3185.51$
2. $I = \$3185.51 - \$3000 = \$185.51$
3. $\$185.51 = (3000)(r)(1)$ $r = \frac{185.51}{3000} = 0.0618$ $\boxed{6.18\%}$

Present Value Formulas

The present value P of A dollars to be received after t years, assuming a per annum interest rate r is compounded n times per year, is $P = A \cdot \left(1 + \frac{r}{n}\right)^{-nt}$. If the interest is compounded continuously, $P = Ae^{-rt}$.

Example: A zero-coupon (noninterest-bearing) bond can be redeemed in 9 years for \$1200. How much should you be willing to pay for it now if you want a return of $A = 1200$ $t = 9$

a) 7% compounded monthly? $n = 12$

$P = 1200 \cdot \left(1 + \frac{.07}{12}\right)^{-12 \cdot 9}$

$\boxed{P = \$640.28}$

b) 6% compounded continuously?

$P = 1200e^{-.06 \cdot 9}$

$\boxed{P = \$699.30}$

Example: What annual rate of interest compounded annually should you seek if you want to double your investment in 7 years? $n = 1$ $t = 7$ $A = 2P$

$A = P\left(1 + \frac{r}{n}\right)^{nt}$

$2P = P(1 + r)^7$

$\frac{2P}{P} = (1 + r)^7$

$2 = (1 + r)^7$

$\sqrt[7]{2} = 1 + r$

$r = \sqrt[7]{2} - 1$

$r = .1041$

$\boxed{10.41\%}$

Example: Find the time required to double or triple an investment if it earns 5% compounded continuously. $r = .05$

Double:

$A = Pe^{rt}$

$2P = Pe^{.05t}$

$2 = e^{.05t}$

$\ln 2 = .05t$

$t = \frac{\ln 2}{.05}$

$t = \boxed{13.86 \text{ years}}$

Triple:

$3P = Pe^{.05t}$

$3 = e^{.05t}$

$\ln 3 = .05t$

$t = \frac{\ln 3}{.05}$

$t = \boxed{21.97 \text{ years}}$

Exponential Growth and Decay Models

Law of Unhibited Growth

Many natural phenomena have been found to follow the law that an amount A varies with time t according to the function $A(t) = A_0 e^{kt}$, where A_0 is the original amount at time $t = 0$ and k is a constant of growth or decay (growth if $k > 0$, decay if $k < 0$.)

Example: The number N of bacteria present in a culture at time t hours obeys the law of uninhibited growth where $N(t) = 1000e^{0.01t}$.

a) Determine the number of bacteria at $t = 0$ hours.

$N(0) = 1000e^{0.01(0)} = \boxed{1000}$

b) What is the growth rate of the bacteria?

0.01/hr or $\boxed{\text{1\% per hour}}$

c) What will the population be after 4 hours?

$N(4) = 1000e^{0.01(4)} = 1040.81$

$\boxed{\text{About 1041 bacteria}}$

d) When will the number of bacteria reach 1700?

$1700 = 1000e^{.01t}$

$1.7 = e^{.01t}$

$\ln 1.7 = .01t$

$t = \dfrac{\ln 1.7}{.01} = \boxed{53.1 \text{ hrs}}$

e) When will the number of bacteria double?

$2000 = 1000e^{.01t}$

$2 = e^{.01t}$

$\ln 2 = .01t$

$t = \dfrac{\ln 2}{.01} = \boxed{69.3 \text{ hrs}}$

Example: The annual growth rate of the world's population in 2005 was $k = 1.15\% = 0.0115$. The population of the world in 2005 was 6,451,058,790 people. Letting $t = 0$ represent the year 2005, use the uninhibited growth model to predict the world's population in the year 2015. In 2015, $t = 10$

$A(t) = A_0 e^{kt}$

$A(t) = 6{,}451{,}058{,}790\, e^{0.0115t}$

$A(10) = 6{,}451{,}058{,}790\, e^{(0.0115 \cdot 10)}$

$= \boxed{7{,}237{,}271{,}501}$

Example: Iodine 131 is a radioactive material that decays according to the function $A(t) = A_0e^{-0.087t}$, where A_0 is the initial amount present and A is the amount present at time t (in days). Assume that a scientist has a sample of 100 grams of iodine 131. $A_0 = 100$

a) What is the decay rate of iodine 131?

$k = -0.087 = \boxed{-8.7\%}$

b) How much iodine 131 is left after 9 days?

$A(9) = 100\, e^{(-0.087)(9)} = \boxed{45.7\text{ g}}$

c) When will 70 grams of iodine 131 be left?

$70 = 100\, e^{-0.087t}$

$.7 = e^{-0.087t}$

$\ln .7 = -0.087t$

$t = \dfrac{\ln .7}{-0.087} = \boxed{4.1\text{ days}}$

d) What is the half-life of iodine 131? (when $A = \frac{1}{2}A_0$.)

$50 = 100\, e^{-0.087t}$

$.5 = e^{-0.087t}$

$\ln .5 = -0.087t$

$t = \dfrac{\ln .5}{-0.087} = \boxed{7.97\text{ days}}$

Example: A piece of charcoal contains 30% of the carbon 14 that it originally had. When did the tree die from which the charcoal came? Use 5600 years as the half-life of carbon 14.

$A = A_0e^{kt}$

$.5A_0 = A_0e^{5600k}$

After 5600 yrs, half of the original material remains.

$.5 = e^{5600k}$

$\ln .5 = 5600k$

$k = \dfrac{\ln .5}{5600} = -.0001238$

$A = A_0e^{-.0001238t}$

$.3A_0 = A_0e^{-.0001238t}$

$.3 = e^{-.0001238t}$

$\ln .3 = -.0001238t$

$t = \dfrac{\ln .3}{-.0001238} = \boxed{9727\text{ years}}$

Example: At 45°C, dinitrogen pentoxide decomposes into nitrous dioxide and oxygen according to the law of uninhibited decay. An initial amount of 0.25 M of dinitrogen pentoxide decomposes to 0.15 M in 17 minutes. How much dinitrogen pentoxide will remain after 30 minutes? How long will it take until 0.01 M of dinitrogen pentoxide remains? $A_0 = 0.25$

$A = A_0e^{kt}$

$.15 = .25e^{k(17)}$

$.6 = e^{17k}$

$\ln .6 = 17k$

$k = \dfrac{\ln .6}{17} = -0.0300$

$A(t) = .25e^{-.03t}$

$A(30) = .25e^{(-.03)(30)}$

$= \boxed{0.102\text{ M}}$

$0.01 = 0.25e^{-.03t}$

$0.04 = e^{-.03t}$

$\ln 0.04 = -0.03t$

$t = \dfrac{\ln 0.04}{-0.03}$

$t = \boxed{107.3\text{ min}}$